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FIXED POINT THEOREMS IN GENERALIZED METRIC SPACES =>

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ABSTRACT

Obtain fixed point theorems by using generalized metric spaces.

KEYWORDS: *Metric Space, b-Metric Spaces, Partial Metric Spaces, Cone Metric Spaces, Generalized Metric Spaces*

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1. INTRODUCTION

In 1922, The great mathematician Banach[4] gives fixed point theorem in complete metric space. Further many mathematician gives fixed point theorems in metric space. Bakhtin [5]gives fixed point theorem in partial metric space. After his result many author gives fixed point results in this space (see 2,3, 8, 16 17,18) and many authors gives fixed point results in b- metric space..(see 6, 9 .10).

On the other hand Abdeljawad and Karapinar [1] gives fixed point theorems in tvs cone metric space and many authors generalized the results of this tvs cone metric space (see 11,14).

Recently, Xun and Songlin[20] proved fixed point theorem in Banach contraction and Kannan contraction on generalized metric space. In this paper obtain fixed point theorems in ciric contraction and Singh contraction type on generalized metric space. Our theorem is generalization the theorem of [7],[12],[13], [15],[19] and others.

2. Priliminaries

Definition 2.1 [20] Let be a topological vector space with zero vector Θ. A subset £ in is called a tvs- cone in , if the following conditions are satisfied .

1. £ is nonempty and closed in .
2. u, v ε £ and l, m ε [0, +∞) imply lu + mv ε £.
3. u, -u ε £ imply u = Θ. .

Definition 2.2 [20] Let £ be a tvs- cone in a topological vector space and P^0 denote the integer of £ in . partial orderings ≤, < and << on with respect to £.. Let u, v ε .

1. u ≤ v if v – u ε £.
2. u < v if u ≤ v and u ≠ v.
3. u < < v if v - u ε P^0.

4. Then (, £) is called an ordered topological vector space.

Definition 2.3 [9] Let M be nonempty set. A function $\zeta : M\times M \to [0,\infty)$ is called b- metric space with coefficient $p \geq 1$ if satisfied the following conditions for u, v, w ε

1. $\zeta(u, v) = 0$ iff $u = v$
2. $\zeta(u, v) = \zeta(v, u)$
3. $\zeta(u, v) \leq p[\zeta(u, w) + \zeta(w, v)]$

Definition 2.4 [6] Let M be nonempty set. A function $\zeta : M\times M \to [0,\infty)$ is called partial metric space if satisfied the following conditions for u, v, w ε M

1. $u = v$ iff $\zeta(u, u) = \zeta(v, v) = \zeta(u, v)$
2. $\zeta(u, v) = \zeta(v, u)$
3. $\zeta(u, u) \leq \zeta(u, v)$
4. $\zeta(u, w) \leq \zeta(u, v) + \zeta(v, w) - \zeta(v, v)$.

Definition 2.5 [15] Let **M** be a nonempty set and (, £) be an ordered topological vector space with its zero vector Θ. A function $\zeta : M \times M \to £$ is called a generalized metric space with co- efficient $p \geq 1$ if the following conditions are satisfied for all u, v, w ε M.

1. $u = v$ iff $\zeta(u, u) = \zeta(v, v) = \zeta(u, v)$
2. $\zeta(u, v) = \zeta(v, u)$
3. $\zeta(u, u) \leq \zeta(u, v)$
4. $\zeta(u, w) \leq p[\zeta(u, v) + \zeta(v, w) - \zeta(v, v)]$.

Definition 2.6 [14] Let **M** be a nonempty set and (, £) be an ordered topological vector space with its zero vector θ. A function $\zeta : M \times M \to £$ is called a tvs – cone metric space if the following conditions are satisfied for all u, v, w ε M.

1. $\zeta(u, v) = \Theta$ iff $u = v$
2. $\zeta(u, v) = \zeta(v, u)$
3. $\zeta(u,w) \leq \zeta(u, v) + \zeta(v, w)$

Lemma 2.7 [14] Let (, £) be an ordered topological vector space.

1. If Θ ε £ – $£^0$ implies $£^0 \neq \Theta$.
2. u, u1, u2, … un ε , u ≤ max { u, u1, u2, … un } denote u ≤ ui for some i = 1, 2, … n.
3. The use notation ≥ , > and >> in (, £). These notation are clear and hold the following,

i. $u \geq v$ iff $u - v \geq \Theta$ iff $u - v$ ε £.

ii. $u > v$ iff $u - v > \Theta$ iff $u - v$ ε £ – { Θ }.

iii. $u >> v$ iff $u - v >> \Theta$ iff $u - v$ ε $£^0$.

iv. u >> v implies u > v implies u≥ v,

Lemma 2.8[14] Let (,£) be an ordered topological vector space.

1. If u >> ϴ , then ku >> ϴ, for each k ε R^+.
2. If u >> ϴ , then u >> 1/2 u >> 1/3 αu>>… >> ϴ
3. If u1 >> v1 and u2 ≥ v2, then u1 + u2 >> v1 + v2 .
4. u >> v ≥ μ or u ≥ v >> μ, then u>> μ.
5. If u >> ϴ and v ε , then n ε N such that ½ v << u.
6. If u >> ϴ and If v >> ϴ, then μ >>ϴ such that μ <<u and μ <<v.

Definition 2.9 [20] Let (, £) be an ordered topological vector space, {u_n} be a sequence in and u ε . {u_n} is called to converges to u in (, £) if for any ε >> ϴ, there is n_0 ε N such that u – v<< u_n << u +ε for all n > n_0, we denoted by **lim**$_{n\to\infty}u_n$ = u

Lemma 2.10 [20] Let (, £) be an ordered topological vector space, {un} be a sequence in and u ε If $\lim_{n\to\infty} u_n = u$, then **lim**$_{n\to\infty}$un = u.

Lemma 2.11 [20] Let (, £) be an ordered topological vector space, {u^n} and {v_n} be a sequence in . If **lim**$_{n\to\infty}$ u_n = u and **lim**$_{n\to\infty}$vn = v. Then **lim**$_{n\to\infty}$ **(**$u_n \pm v_n$) = u ± v.

Lemma 2.12 [20] Let (, £) be an ordered topological vector space, {un} and {v_n} be a sequence in E. Then

i. Let un ≥ v_n for all n ε N. If $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$ then u ≥ v.

ii. Let un ≥ v_n ≥ μ_n for all n ε N. If $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \mu_n = u$, then $\lim_{n\to\infty}$vn = u.

Definition 2.13 [20] Let (Ϻ, ζ) be a generalized metric space. A sequence {un} in Ϻ is said to converges to u if for any ε >> ϴ such that ζ (u, un) << ζ (u, u) + ε for all n > n0, n ε N so, limn→∞un = u.

Definition 2.14 [20] Let (Ϻ, ζ) be a generalized metric space. A sequence {u_n} in Ϻ and u ε Ϻ. Then both are equivalent.

1. limn→∞un = u.
2. **limn**→∞ ζ (u, un) = ζ (u, u).

Definition 2.15 [20] Let (Ϻ, ζ) be a generalized metric space. A sequence {u_n} in Ϻ

1. {un} is Cauchy sequence in (Ϻ, ζ) if u ε Ϻ, such that limn,m→∞ ζ (un, um) = u ie, for any ε >> ϴ, there is n0 ε N such that u - v << ζ (un, um) << u + ε for all n, m > n0.
2. (Ϻ, ζ) is complete if for each Cauchy sequence {un} is convergent, if u ε Ϻ such that ζ (u, u) = **limn,m**→∞ ζ (un, um) = **limn,m**→∞ ζ (un, um).

Definition 2.16 [20] Let (Ϻ, ζ) be a generalized metric space with coefficients p ≥ 1 as ɧ : Ϻ → Ϻ be a function u ε Ϻ is called a fixed point of ɧ if ɧu = u. We denote the set of fixed point of ɧ by Fix(ɧ) and cardinal of Fix (ɧ) by I Fix(ɧ) I.

[20] proved the following fixed point theorem.

Theorem 2.17 [20] Let (M, ζ) be a complete generalized metric space with co- efficient p ≥ 1 and let ђ : M → M be a function such that

ζ(ђu, ђv) ≤ g ζ (u, v) for all u, v ε M, where g ε [0, 1) and gp < 1. Then ђ has a unique fixed point u ε M and ζ(u, u) = Θ.

Theorem 2.18 [20] Let (M, ζ) be a complete generalized metric space with cefficient s ≥ 1 and let ђ : M → M be a function such that

ζ(ђu, ђv) ≤ g[ζ (u, ђu)+ζ (v, ђv)] for all u, v ε M, where k ε [0, 1) and gp < 1. Then ђ has a unique fixed point u ε M and ζ (u, u) = Θ.

Theorem 2.19 [20] Let (M, ζ) be a complete generalized metric space with coefficient s ≥ 1 and let ђ : M→ M be a function such that

ζ (ђu, ђv) ≤ g max{ ζ (u, v), ђ (u, ђu), ђ (v, ђv) } for all u, v ε M, where g ε [0, 1) and gp < 1.Then ђ has a unique fixed point u ε M and ζ (u, u) = Θ.

Theorem 2.20 [20] Let (M, ζ) be a complete generalized metric space with co efficient p ≥ 1 and let ђ : M → M be a function such that

ζ (ђu, ђv) ≤ g1 ζ (u, v) + g2 ζ (u, ђu) + g3 ђ (v, ђv), for all u, v ε M, where g1 + g2 + g3 ε [0, 1/p).Then ђ has a unique fixed point u ε M and ζ(u, u) = Θ.

Lemma 2.21 [20] Let (, £) be a ordered topological vector space and ε ε $£^0$. Put u = R^*, where R^* is the set of all non-negative real number, for n ε N, define

ζn : M × M → £ by

ζn(u, v) = ({ max (u, v) $\}^n$ + I u – v I^n)ε. Then (M, ζn) is a generalized metric space with co- efficient p = 2^{n-1}.

3. Main Results

Our main results is following theorems and examples.

Theorem 3.1 Let (M, ζ) be a complete generalized metric space with coefficient p ≥ 1 and let ђ : M → M be a function such that

ζ (ђu, ђv) ≤ g max{ ζ (u, v), ζ (u, ђu), ζ (v, ђv), ζ (u, ђv), ζ (v, ђu) } for all u, v ε M, where g ε [0, 1/p). Then ђ has a unique fixed point u ε M and ζ (u, u) = Θ.

Proof- Proof the theorem by two claim.

Claim 1. If Fix (ђ) ≠ Θ, then I Fix(ђ) I = 1.

Let Fix (ђ) ≠ Θ. If u, v ε Fix (ђ) , that is u, v ε M, ђu = u and ђv = v, then

ζ (u, v) = ζ(ђu, ђv) ≤ g max{ ζ (u, v), ζ(u, ђu), ζ(v, ђv), ζ (u, ђv), ζ(v, ђu) }

= g max{ d (u, v), ζ (u,u), ζ (v, v), ζ(u, v), ζ (v, u) }

$= g\,\zeta(u, v)$

Implies , $\zeta(u,v) = 0$, since $g < 1$, then I Fix(T) I = 1.

Claim 2. If u ε Fix (ɦ) such that $\zeta(u, v) = \Theta$.

Let u0 ε Ϻ and un = ɦun-1 for each n ε N and i, j ε N and $i \neq j$,

$u_i \neq u_j$ and so $\zeta(u_i, u_j) > \Theta$. For each n ε N,

$\zeta(u_n, u_{n+1}) = \zeta(ɦu_{n-1}, ɦu_n) \leq g \max\{\zeta(u_{n-1}, u_n), \zeta(u_{n-1}, u_n), \zeta(u_n, u_{n+1}), \zeta(u_{n-1}, u_{n+1}), \zeta(u_n, u_n)\}$

$= g \max\{\zeta(u_{n-1}, u_n), \zeta(u_n, u_{n+1})\}$

This implies that

$\zeta(u_n, u_{n+1}) \leq g\,\zeta(u_n, u_{n+1})$ or $\zeta(u_n, u_{n+1}) \leq k\,\zeta(u_{n-1}, u_n)$.

If $\zeta(u_n, u_{n+1}) \leq g\,\zeta(u_n, u_{n+1})$, then $\zeta(u_n, u_{n+1}) < \zeta(u_n, u_{n+1})$.This is a contradiction. So, $\zeta(u_n, u_{n+1}) \leq g\,\zeta(u_{n-1}, u_n)$

$\leq g^n \zeta(u_o, u_1)$.

Let n, m ε N, then

$0 \leq \zeta(u_n, u_{n+m}) \leq p\,\zeta(u_n, u_{n+1}) + p^2 \zeta(u_{n+1}, u_{n+2}) + \ldots + p^m \zeta(u_{n+m-1}, u_{n+m})$

$\leq p\,g^n \zeta(u_0, u_1) + p^2 g^{n+1} \zeta(u_0, u_1) + \ldots + p^m g^{n+m-1} \zeta(u_0, u_1)$.

$\leq [p\,g^n + p^2 g^{n+1} + \ldots + p^m g^{n+m-1}]\,\zeta(u_0, u_1)$.

$\leq p\,g^n [1 + p + \ldots + p^{m-1} g^{m-1}]\,\zeta(u_0, u_1)$.

$= p\,g^n \zeta(u_0, u_1)/ 1- pg$.

Since, $0 \leq k \leq pg \leq 1$, $\lim_{n\to\infty} pg^n/1- pg = 0$, hence $\lim_{n\to\infty} pg^n \zeta(u_0, u_1)/1- pg = \Theta$.

By lemma2.12, $\mathbf{\lim_{n,m\to\infty}} \zeta(u_n, u_m) = \Theta$. So {un} is a Cauchy sequence in (Ϻ, ζ). Since (Ϻ, ζ) is complete, hence

$\zeta(u, u) = \mathbf{\lim_{n\to\infty}} \zeta(u, u_n) = \mathbf{\lim_{n,m\to\infty}} \zeta(u_n, u_m) = \Theta$.

$\zeta(u_n, ɦu) = \zeta(ɦu_{n-1}, ɦu) \leq g \max\{\zeta(u_{n-1}, u), \zeta(u_{n-1}, u_n), \zeta(u, ɦu), \zeta(u, u_n), \zeta(u_{n-1}, ɦu)\}$

$\zeta(u, ɦu) \leq p\,(\zeta(u, u_n) + \zeta(u_n, ɦu))$

$\leq p\zeta(u, u_n) + g \max\{\zeta(u_{n-1}, u), \zeta(u_{n-1}, u_n), \zeta(u, ɦu), \zeta(u, u_n), \zeta(u_{n-1}, ɦu)\}$.

By lemma2.11, $\zeta(u, ɦu) \leq gp\,\zeta(u, ɦu)$. $0 < gp < 1$, $\zeta(u, ɦu) = \Theta$. So, u = ɦu.

Hence, u ε Fix (ɦ) and $\zeta(u,v) = \Theta$.

An application of Theorem 3.1 , the following corollary generalizes a fixed point theorem of [12].

Corollary 3.2 Let (M, ζ) be a complete generalized metric space with co efficient $p \geq 1$ and let ƥ : M → M be a function such that

ζ (ƥu, ƥv) ≤ g1 ζ (u, v) + g2 ζ (u, ƥu) + g3 ζ (v, ƥv) +g_4 ζ (u, ƥv) + g5 ζ(v, ƥu) for all u, v ε M,

where g1 + g2 + g3 + g4 + g5 ε [0. 1/p). Then ƥ has a unique fixed point u ε M and ζ (u, u) = Θ.

Proof- Put g = g1 + g2 + g3 + g4 + g5, then g ε [0. 1/p). For all x, y ε X,

ζ (ƥu, ƥv) ≤ g1 ζ (u, v) + g2 ζ (u, ƥu) + g3 ζ (v, ƥv) +g_4 ζ (u, ƥv) + g5 ζ(v, ƥu)

≤ g1 max{ ζ (u, v), ζ (u, ƥu), ζ (v, ƥv), ζ (u, ƥv), ζ (v, ƥu)} + g2 max{ ζ (u, v), ζ (u, ƥu), ζ (v, ƥv),

ζ (u, ƥv), ζ (v, ƥu) }+ g3 max{ ζ (u, v), ζ (u, ƥu),ζ(v, ƥv), ζ (u, ƥv), ζ (u, ƥu) } + g4 max{ ζ (u, v), ζ (u, ƥu), ζ (

v, ƥv), ζ (u, ƥv), ζ (v, ƥu) } + g5 max{ ζ (u, v), ζ (u, ƥu.), ζ (v, ƥv), ζ (u, ƥv), ζ (v, ƥu) }

=(g1 + g2 + g3 + g4 + g5) max{ ζ (u, v), ζ (u, ƥu), ζ (v, ƥv), ζ (u, ƥv), ζ (v, ƥu)}

= g max{ ζ (u, v), ζ (u, ƥu), ζ (v, ƥv), ζ (u, ƥv), ζ(v, ƥu) }.

Theorem 3.1, ƥ has a unique fixed point u ε M and ζ (u, u) = Θ.

The following Theorem generalizes a fixed point theorem of [4].

Theorem 3.3 Let (M, ζ) be a complete generalized metric space with co- efficient $p \geq 1$ and let ƥ : M → M be a function such that

ζ(ƥu, ƥv) ≤ g max{ ζ (u, v), ζ(u, ƥu), ζ(v, ƥv), 1/2[ζ (u, ƥv)+ ζ(v, ƥu)] } for all u, v ε M, where k ε [0, 1/p).

Then ƥ has a unique fixed point u ε M and ζ (u, u) = Θ.

Proof- proof of theorem by two claims.

Claim 1. If Fix (ƥ) ≠ Θ, then I Fix(Θ) I = 1.

Let Fix (ƥ) ≠ Θ. If u, v ε Fix (ƥ) that is u, v ε X, ƥu = u and ƥv = v, then

ζ (u, v) = ζ(ƥu, ƥv) ≤ g max{ ζ (u, v), ζ(u, ƥu), ζ(v, ƥv), 1/2[ζ (u, ƥv)+ ζ(v, ƥu)] }

= g max{ d (u, v), ζ (u,u), ζ (v, v), 1/2[ζ(u, v)+ ζ (v, u)] }

= g ζ (u, v)

Implies , ζ (u,v) = 0 , since g < 1 , then I Fix(T) I = 1.

Claim 2. If u ε Fix (ƥ) such that ζ (u, v) = Θ.

Let u0 ε M and un = ƥun-1 for each n ε N and i, j ε N and i ≠ j,

ui ≠ uj and so ζ (ui, uj) > Θ. For each n ε N,

ζ (un, un+1) = ζ(ђun-1, ђun) $\leq$ g max { ζ un-1, un), ζ (un-1, un) , ζ (un, un+1),1/2 [ζ(un-1, un+1)+ ζ (un, un)] } = g max { ζ (un-1, un), ζ (un, un+1) }

This implies that

ζ(un, un+1) $\leq$ g ζ (un, un+1) or ζ (un, un+1) $\leq$ g ζ (un-1, un).

If ζ (un, un+1) $\leq$ g ζ (un, un+1), then ζ (un, un+1) < ζ (un, un+1).This is a contradiction. So, ζ(un, un+1) $\leq$ g ζ (un-1, un)

$\leq g^n$ ζ (uo, u1)

Let n, m ε N, then

0 $\leq$ ζ (un, un+m) $\leq$ p ζ (un, un+1) + p^2 ζ (un+1, un+2) + … + p^m ζ (un+m-1, un+m)

$\leq$ p g^n ζ (u0, u1) + p^2 g^{n+1} ζ (u0, u1) + … + p^m g^{n+m-1} ζ (u0, u1).

$\leq$ [p g^n + p^2 g^{n+1} + … + p^m g^{n+m-1}] ζ (u0, u1).

$\leq$ p g^n [1 + p + … + p^{m-1} g^{m-1}] ζ(u0, u1).

= p g^n ζ (u0, u1)/ 1- pg .

Since, 0 $\leq$ g $\leq$ gk $\leq$ 1, limn$\to\infty$ pg^n/1- pg = 0, hence limn$\to\infty$ pg^n ζ (u0, u1) /1- pg = ϴ.

By lemma2.12, **limn,m**$\to\infty$ ζ (un, um) = ϴ. So {un} is a Cauchy sequence in (Ϻ, ζ). Since (Ϻ, ζ) is complete, hence

ζ (u, u) = **limn**$\to\infty$ ζ (u, un) = **limn,m**$\to\infty$ ζ (un, um) = ϴ.

ζ (un, ђ u) = ζ(ђun-1, ђu) $\leq$ g max { ζ(un-1, u), ζ (un-1, un) , ζ (u, ђu), 1/2[ζ (u, un)+ ζ (un-1, ђ u)] }

ζ (u, ђu) $\leq$ p (ζ (u, un) + ζ (un, ђ u))

$\leq$ pζ (u, un) + g max { ζ (un-1, u), ζ (un-1, un), ζ (u, ђu),1/2[ζ (u, un)+ ζ (un-1, ђ u)] }.

By lemma2.11, ζ (u, ђu) $\leq$ gp ζ (u, ђu). 0 < gp < 1, ζ(u, ђu) = ϴ. So, u = ђu.

Hence, u ε Fix (ђ) and ζ (u,v) = ϴ.

An application of Theorem 3.3 , the following corollary generalizes a fixed point theorem of [19].

Corollary 3.4 Let (Ϻ, ζ) be a complete generalized metric space with co efficient p $\geq$ 1 and let ђ : Ϻ → Ϻ be a function such that

ζ (ђu, ђv) $\leq$ g1 ζ (u, v) + g2 ζ (u, ђu) + g3 ζ (v, ђv) +g_4 [ζ (u, ђv) + ζ(v, ђu)] for all u, v ε Ϻ,

where g1 + g2 + g3 +2 g4 ε [0. 1/p). Then ђ has a unique fixed point u ε Ϻ and ζ (u, u) = ϴ.

Proof- Put k = k1 + k2 + k3 +2 k4 , then g ε [0. 1/p). For all u, v ε Ϻ,

ζ (ђu, ђv) $\leq$ g1 ζ (u, v) + g2 ζ (u, ђu) + g3 ζ (v, ђv) +$g_{4[}$ ζ (u, ђv) + ζ(v, ђu)]

≤ g1 max{ ζ (u, v), ζ (u, ħu), ζ (v, ħv),1/2[ζ (u, ħv), ζ (v, ħu)]} + g2 max{ ζ (u, v), ζ (u, ħu), ζ (v, ħv),1/2[ζ (u, ħv)+ ζ (v, ħu)] }+ g3 max{ ζ (u, v), ζ (u, ħu),ζ(v, ħv),1/2[ζ (u, ħv)+ ζ (u, ħu)] } + 2g4 max{ ζ (u, v), ζ (u, ħu), ζ (v, ħv),1/2[ζ (u, ħv), ζ (v, ħu)] }

=(g1 + g2 + g3 +2 g4) max{ ζ (u, v), ζ (u, ħu), ζ (v, ħv)1/2,[ζ (u, ħv)+ ζ (v, ħu)]}

= g max{ ζ (u, v), ζ (u, ħu), ζ (v, ħv),1/2[ζ (u, ħv)+ζ(v, ħu)] }.

Theorem 3.3, ħ has a unique fixed point u ε M and ζ (u, u) = Θ.

Example 3.5. Let = { (u, v) : u, v ε R } and £ = { (u, v) : u, v ε R^* }. Then (, £) is an ordered topological vector space. Let M = { 0, 1, 2 }.

Define a function

ζ : M × M→ by

ζ(u, v) = ({ max (u, v) $\}^2$ + I u – v I^2)ε, where Ω = (1,1) ε $£^0$. Then

ζ (0, 0) = Θ, ζ (1,1) = Ω, ζ (2,2) = 4 Ω, ζ (0,1) = 2 Ω, ζ (0,2) = 8 Ω, ζ (1,2) = 5Ω. Define a function

ħ : M → M by

ħ0 = ħ1 = 0 and ħ2 = 1 then

ζ(ħ0, ħ0) = ζ (0, 0) = θ, ζ (ħ1, ħ1) = ζ (0, 0) = θ, ζ (ħ2, ħ2) = ζ (1, 1) = Ω, ζ (ħ0, ħ1) = ζ (0, 0) = θ,ζ (

ħ0, ħ2) = ζ (0, 1) = 2Ω, ζ(ħ1, ħ2) = ζ (0, 1) = 2Ω, ζ (0, ħ0) = ζ (0, 0) = θ, ζ (1, ħ1) = ζ (1, 0) = 2Ω and ζ (2, ħ2) = ζ (2, 1) = 5Ω.

By lemma2.21, (M, ζ) is a generalized metric space with coefficient p = 2^{2-1} = 2, which is a partial tvs- cone metric space. Obviously (M, ζ) is complete.

max{ ζ (0, 0), ζ (0, ħ0), ζ (0, ħ0), ζ (0, ħ0), ζ (0, ħ0) } = Θ

max{ ζ (1, 1), ζ (1, T1), ζ(1, T1), ζ (1, T1), ζ (1, T1) } = max{ ζ (1, 1), ζ(1, 0), ζ (1, 0), ζ (1, 0), ζ (1, 0) } = 2Ω

max{ ζ (2, 2), ζ (2, ħ2), ζ(2, ħ2), ζ (2, ħ2), ζ (2, ħ2) } = max{ ζ (2, 2), ζ(2, 1), ζ(2, 1), ζ (2, 1), ζ (2, 1) } = 5Ω

max{ ζ (0, 1), ζ(0, ħ0), ζ (1, ħ1), ζ (0, ħ1), ζ (1, ħ0) } = max{ ζ (0, 1), ζ (0, 0), ζ (1, 0), ζ (0, 0), ζ (1, 0) } = 2Ω

max{ ζ (0, 2), ζ (0, ħ0), ζ (2, ħ2), ζ (0, ħ2), ζ (2, ħ0) } = max{ ζ (0, 2), ζ(0, 0), ζ (2, 1), ζ (0, 1), (2, 0) } = 8Ω

max{ ζ (1, 2), ζ (1, ħ1), ζ (2, ħ2), ζ(1, ħ2), ζ (2, ħ1) } = max{ ζ (1, 2), ζ(1, 0), ζ(2, 1), ζ (1, 1), ζ (2, 0) } =

5Ω.

It is easily to check that

ζ (ƕu, ƕv) ≤ 2/5 max{ ζ(u, v), ζ (u, ƕu), ζ (v, ƕv), ζ (u, ƕv), ζ (v, ƕu) } for all u, v ε M. In addition

2/5 ε [0. 1/p), since p = 2. So verify Theorem 2.1.

≤ p ζ (u, /un) + k max { ζ (un-1, u), ζ (un-1, un), ζ(u, ƕu),1/2 [ζ (u, un)+ ζ (un-1, ƕ u)] }.

By lemma2.11 and so ζ (u, ƕu) ≤ gp ζ (u, ƕu). Since gp < 1, ζ (u, ƕu) = Θ. So, u = ƕu. This implies that u ε Fix (ƕ) and ζ (u, v) = Θ.

Example 3.6. Let = { (u, v) : u, v ε R } and £ = { (u, v) : u, v ε R^* }. Then (, £) is an ordered topological vector space. Let M = { 0, 1, 2 }.

Define a function

ζ : M × M → by

ζ(u, v) = ({ max (u, v) } + I u – v I)ε, where Ω= (1,1) ε $£^0$. Then

ζ (0, 0) = Θ, ζ (1,1) = Ω, ζ (2,2) = 2 Ω, ζ (0,1) = 2 Ω, ζ (0,2) = 4 Ω, ζ (1,2) = 3Ω. Define a function

ƕ : M → M by

ƕ0 = ƕ1 = 0 and ƕ2 = 1 then

ζ(ƕ0, ƕ0) = ζ (0, 0) = Θ, ζ (ƕ1, ƕ1) = ζ (0, 0) = Θ, ζ (ƕ2, ƕ2) = ζ (1, 1) = Ω, ζ (ƕ0, ƕ1) = ζ (0, 0) = Θ,ζ (

ƕ0, ƕ2) = ζ (0, 1) = 2Ω, ζ(ƕ1, ƕ2) = ζ (0, 1) = 2Ω, ζ (0, ƕ0) = ζ (0, 0) = θ, ζ (1, ƕ1) = ζ (1, 0) = 2Ω and ζ (2, ƕ2) = ζ (2, 1) = 3Ω.

By lemma2.21,(M, ζ) is a generalized metric space with coefficient p = 2^{1-1} = 1 , which is a partial tvs- cone metric space. Obviously (M, ζ) is complete.

max{ ζ (0, 0), ζ (0, ƕ0), ζ (0, ƕ0),1/2[ζ (0, ƕ0), ζ (0, ƕ0)[} = Θ

max{ ζ (1, 1), ζ (1, ƕ1), ζ(1, ƕ1),1/2[ζ (1, ƕ1)+ ζ (1, ƕ1)] } = max{ ζ (1, 1), ζ(1, 0), ζ (1, 0),1/2[ζ (1, 0)+ ζ(1, 0)] } = 2Ω

max{ ζ (2, 2), ζ (2, ƕ2), ζ(2, ƕ2),1/2[ζ (2, ƕ2)+ ζ (2, ƕ2)] } = max{ ζ (2, 2), ζ(2, 1), ζ(2, 1),1/2[ζ (2, 1)+ ζ (2, 1)] } =3Ω

max{ ζ (0, 1), ζ(0, ƕ0), ζ (1, ƕ1)1/2,[ζ (0, ƕ1)+ ζ (1, ƕ0)] } = max{ ζ (0, 1), ζ (0, 0), ζ (1, 0)1/2,[ζ (0, 0)+ ζ (1, 0)] } =2Ω

max{ ζ (0, 2), ζ (0, ƕ0), ζ (2, ƕ2),1/2[ζ (0, ƕ2)+ ζ (2, ƕ0)] } = max{ ζ (0, 2), ζ(0, 0), ζ (2, 1)1/2,[ζ (0, 1)+ (2, 0)]} =4Ω

max{ ζ (1, 2), ζ (1, ƕ1), ζ (2, ƕ2)1/2,[ζ(1, ƕ2) +, ζ (2, ƕ1)] } = max{ ζ (1, 2), ζ(1, 0), ζ(2, 1),1/2[ζ (1, 1)+ ζ (2, 0)] } =3Ω.

It is easily to check that

ζ (ḫu, ḫv) ≤ 2/3 max{ ζ(u, v), ζ (u, ḫu), ζ (v, ḫv), [ζ (u, ḫv)+ ζ (v, ḫu)] } for all u, v ε M. This implies

2/3 ε [0. 1/p), since p = 1. So verify Theorem 3.3.

≤ p ζ (u, un) + g max { ζ (un-1, u), ζ (un-1, un), ζ(u, ḫu),1/2 [ζ (u, un)+ ζ (un-1, ḫ u)] }.

By lemma1.11 and so ζ (u, ḫu) ≤ gp ζ (u, ḫu). Since gp < 1, ζ (u, ḫu) = Θ. So, u = ḫu. This implies that u ε Fix (ḫ) and ζ (u, v) = Θ.

Remarks 2.7.

1. Theorem 3.1 and 3.3 give generalizations of theorem[3], [4] and [7].
2. Theorem 3.1 and 2.3 give generalizations of Theorem 3.5, 3.6 and 3.7 of [20].

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